Existence and Uniqueness of Solutions for Non-Autonomous Complementarity Dynamical Systems

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This paper deals with the well-posedness of a class of complementarity dynamical systems. Both the linear and the nonlinear cases are treated, and the systems are non-autonomous. A specific "input-output" property is used to perform a change of state vector which allows one to transform the complementarity dynamics into a perturbed Moreau's sweeping process. Then the results obtained elsewhere by Thibault and his co-workers [16, 15, 40] on the well-posedness of the sweeping process are used. Absolutely continuous as well as bounded variation solutions (with state jumps) are examined in this work.

Keywords: Moreau's sweeping process, complementarity system, differential inclusion, existence, uniqueness, prox-regular set, state jumps, bounded variation

1. Introduction

Complementarity systems have recently been the object of strong interest because of their applications in various fields like mechanics, electrical circuits, transportation science, control systems, etc, see for instance [20, 10, 7, 3, 43, 17, 19, 6]. The relationships between complementarity systems and other formalisms have been studied in [8, 22]. Related results are in [20, 21] where it is considered a non-autonomous extension of Kato's theorem for maximal monotone variational inequalities. In this paper we are interested in analysing the existence and uniqueness of solutions of two classes of complementarity systems, by embedding their dynamics into the framework of so-called *perturbed sweeping processes*. The sweeping process is a particular differential inclusion that has been introduced by J. J. Moreau [31, 32, 30] and widely studied since then. Its simplest form is

$$-\dot{x}(t) \in N(S(t); x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$. The set $N(S(t); x(t)) \subset \mathbb{R}^n$ is the normal cone to the time-varying set $S(t)$ at the point $x(t)$. When $S(t)$ is nonempty, closed and convex for each t, this is the normal cone of convex analysis, that is defined as [23, 38]:

$$N(S(t); x(t)) = \{z \in \mathbb{R}^n \mid \langle z, y - x(t)\rangle \leq 0 \text{ for all } y \in S(t)\}. \tag{2}$$

In particular one has $N(S(t); x(t)) = \{0\}$ when $x(t)$ is in the interior of $S(t)$, so that (1) reduces to $\dot{x}(t) = 0$. When $x(t)$ lies on the boundary of $S(t)$, then there exists a (possibly non zero) element $\xi(t)$ of $N(S(t); x(t))$ such that $-\dot{x}(t) = \xi(t)$ (such a mapping $\xi(\cdot)$ is called a *selection* in the theory of differential inclusions). Consequently the only way to make the "point" $x(\cdot)$ move is to "sweep" it with the set $S(\cdot)$. The original motivations of the sweeping process were in mechanics (micromechanical damage, quasitatic evolution problems with friction, elastoplasticity, dynamics with unilateral contact, etc). The *perturbed sweeping process* is a differential inclusion of the form

$$-\dot{x}(t) \in N(S(t); x(t)) + f(t, x(t)) \tag{3}$$

where $f(\cdot,\cdot)$ is a vector field, ususally called a perturbation in the mathematical literature. In this paper we shall have to consider a generalization of (2) when $S(t)$ is not convex but is r-prox-regular. A definition of r-prox-regularity is in Section A.2, and the definition of the normal cone to a subset of $\mathbb{R}^n$ is given in Section A.1. Well-posedness results on various forms of the sweeping process are numerous, see e.g., [30, 16, 15, 40, 4, 2, 11, 14, 24, 25, 12, 27]. In particular when the state $x(\cdot)$ suffers from discontinuities, the differential inclusion in (3) has to be rewritten as a *measure differential inclusion*, which is a generalization of (3). Measure differential inclusions were introduced by Moreau in [30]. We shall discuss this more accurately in Section 3.2.

Let us now consider the following complementarity dynamical system

$$\begin{cases} \dot{x}(t) = a(x(t)) + b(x(t))\zeta(t) + e(x(t), u(t)) \\ 0 \leq \zeta(t) \perp w(t) = c(x(t)) + g(u(t)) \geq 0, \end{cases} \tag{4}$$

where $u(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^m$. The second line is a complementarity relation between $w(t)$ and $\zeta(t)$, which are forced to remain always orthogonal one to each other (both inequalities are to be understood componentwise, so that the orthogonality can equivalently be expressed componentwise). This class is very general and needs to be narrowed to obtain well-posedness results. In this paper we shall concentrate our efforts on the case when $b(x) = B$, a constant matrix, and on the so-called linear complementarity systems [10, 42] of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\zeta(t) + Eu(t) \\ 0 \leq \zeta(t) \perp w(t) = Cx(t) + D\zeta(t) + Gu(t) + F \geq 0, \end{cases} \tag{5}$$

where the matrices and vectors A, B, C, D, E, F, G are constant of suitable dimensions, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $\zeta(t) \in \mathbb{R}^m$. More specific assumptions will be introduced later. For instance we shall make the following assumption to perform the study of (5) with $D = 0$:

Assumption 1.1. Let $D = 0$. There exists a constant matrix $P = P^T > 0$ such that

$$PB = C^T. \tag{6}$$

It is noteworthy that despite P is required to be full-rank, no rank assumption is made on B and C, which may be important in view of applications in electrical circuits where often one has $m > n$. A particular case where $D \neq 0$ will be examined also, with a similar assumption to be made. In the nonlinear case the precise statement of the assumption will be made later. As long as only well-posedness is concerned, this "input-output" property will suffice. Dissipative systems (or positive real systems in the linear time-invariant case) satisfy such a property from the well-known Kalman-Yakubovic-Popov Lemma and its various extensions [9]. But it is noteworthy that no stability is required for the matrix A, thus enlarging the class of systems to a much broader class than positive real (or dissipative in the nonlinear case) systems. Roughly speaking, relation (6) will allow us to transform the complementarity system into a *gradient* complementarity system, which in turn will be transformed into a perturbed sweeping process. The next developments then rely on arguments obtained from recent results in [16, 15], and adapted to our case. This work generalizes some results obtained in [5] where the link between Moreau's process and dissipative complementarity systems was pointed out, and a specific state space transformation relying on the "input-output" property (6) has been proposed. It may also be seen as enlarging the studies in [8, 22] on the equivalences between different formalisms like differential inclusions and complementarity systems.

The paper is organized as follows: in the next section two physical examples that fit with the class of nonsmooth systems we deal with are presented. Section 3 is devoted to prove the well-posedness of the linear perturbation case, whereas Section 4 is dedicated to the nonlinear perturbation case. Results and definitions from nonsmooth analysis are recalled in the Appendix. Throughout the paper we shall also recall some definitions and notions that are useful for the developments.

2. Physical examples

Let us consider the electrical circuit with an ideal diode in Figure 2.1,

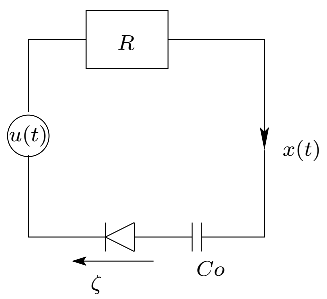

Figure 2.1: A circuit with an ideal diode, a resistor, a capacitor and a voltage source.

with $R > 0$, $Co > 0$. Let $x(\cdot)$ denote the current through the circuit, $z(t) = \int_0^t x(t)dt$, and $\zeta(\cdot)$ the voltage of the diode. The dynamics of this circuit is:

$$\begin{cases} \dot{z}(t) = -\dfrac{u(t)}{R} - \dfrac{1}{RCo} z(t) + \dfrac{1}{R}\zeta(t) \\ 0 \leq \zeta(t) \perp \dfrac{u(t)}{R} - \dfrac{1}{RCo} z(t) + \dfrac{1}{R}\zeta(t) \geq 0. \end{cases} \tag{7}$$

for all $t \geq 0$. As second example let us consider the circuit of Figure 2.2, where $x(\cdot)$ is the current through the inductance, and one considers a current source $i(t)$. The dynamics is given by (notice that the inductance value L does not appear in the dynamics because the set of non negative real numbers $\mathbb{R}_+$ to which ζ belongs is a cone):

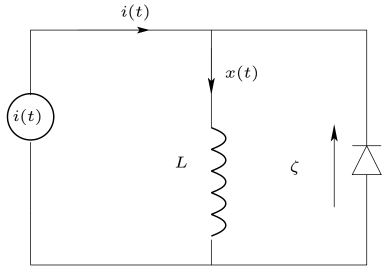

Figure 2.2: A circuit with an ideal diode, an inductor and a current source.

$$\begin{cases} \dot{x}(t) = \zeta(t) \\ 0 \leq \zeta(t) \perp x(t) - i(t) \geq 0. \end{cases} \tag{8}$$

In both examples the matrices A, B, C, D, E, F, G can be easily identified. Other examples may be found in [7, 1, 3, 43, 17, 19]. It is noteworthy that in general one has $D \geq 0$ and non symmetric, see for instance a 4-diode bridge wave rectifier in [1, Chapter 14]. In this paper we shall only treat a particular case of a nonzero matrix D (see Section 3.3). Other fields of application exist in transportation science and macro-economics, as under certain hypotheses complementarity systems are equivalent to projected dynamical systems and various types of differential inclusions and variational inequalities [8].

3. Well-posedness of the dynamics of the LCS

In this section we deal with the linear complementarity system (LCS) in (5). We first treat the case where $D = 0$. Then the case $D \geq 0$ is examined, since it has important practical applications. The case where $D > 0$ (or more generally when D is a P-matrix) is left apart. Indeed the complementarity problem in the second line of (5) is then a linear complementarity problem with a P-matrix, and it has a unique solution. Thus the multiplier ζ is a continuous piecewise linear function of $x(t)$, see Section A.4. Therefore the LCS is nothing else but an ordinary differential equation with a Lipschitz right-hand-side. This is the case of the electrical circuit in Figure 2.1 and dynamics in (7), whose dynamics may be equivalently rewritten as

$$\dot{x}(t) = -\frac{u(t)}{R} - \frac{1}{RCo}x(t) + \text{proj}\left[\mathbb{R}_+; \frac{u(t)}{R} + \frac{1}{RCo}x(t)\right]. \tag{9}$$

On the contrary, the circuit of Figure 2.2 and dynamics in (8) has $D = 0$, and is not represented by an ordinary differential equation, as will be made clear next.

3.1. State space transformation ($D = 0$)

We now employ a state space transformation proposed in [5, 20] which allows us to express linear passive complementarity systems without feedthrough matrix D, into an evolution variational inequality or a differential inclusion. We recall the steps of the transformation for readability sake. In the whole Section 3.1 it is supposed that Assumption 1.1 holds. Defining R as $R^2 = P$, the symmetric positive definite square root of P, and letting $z = Rx$, one gets from (5):

$$\begin{cases} \dot{z}(t) = R\dot{x}(t) = RAR^{-1}z(t) + REu(t) + RB\zeta(t) \\ 0 \le \zeta(t) \perp w(t) = CR^{-1}z(t) + Gu(t) + F \ge 0. \end{cases} \tag{10}$$

Let us assume for the moment that both $\zeta(\cdot)$ and $w(\cdot)$ are functions of time. From a basic result of convex analysis one may write

$$0 \le \zeta(t) \perp CR^{-1}z(t) + Gu(t) + F \ge 0 \Leftrightarrow -\zeta(t) \in \partial\psi_Q(CR^{-1}z(t) + Gu(t) + F)$$

for $Q = \mathbb{R}^m_+$, where $\psi_Q(\cdot)$ denotes the indicator function of the set Q, i.e., $\psi_Q(x) = 0$ if $x \in Q$ and $\psi_Q(x) = +\infty$ otherwise, and ∂ denotes the subdifferential of convex analysis [38, 23]. Consequently one equivalently rewrites (10) as

$$-\dot{z}(t) \in -RAR^{-1}z(t) - REu(t) + RB\,\partial\psi_{\mathbb{R}^m_+}(CR^{-1}z(t) + Gu(t) + F).$$

The equivalence means here that the two formalisms are strictly the same way of writing a mathematical object like a complementarity problem between two variables, without further consideration on the solutions. Now using $R^2B = C^T$ it follows that

$$-\dot{z}(t) \in -RAR^{-1}z(t) - REu(t) + R^{-1}C^T\,\partial\psi_{\mathbb{R}^m_+}(CR^{-1}z(t) + Gu(t) + F). \tag{11}$$

For each $t \in [0, +\infty[$ the closed set

$$K(t) := \{x \in \mathbb{R}^n \mid \; Cx + Gu(t) + F \ge 0\} \tag{12}$$

and $\mathbb{R}^m_+$ are convex polyhedral and $\psi_{K(t)}(x) = (\psi_{\mathbb{R}^m_+ - Gu(t) - F} \circ C)(x)$. Therefore by Proposition A.2 in the Appendix we have

$$C^T\,\partial\psi_{\mathbb{R}^m_+}(Cx + Gu(t) + F) = \partial\psi_{K(t)}(x)$$

for any $x \in \mathbb{R}^n$. So the inclusion in (11) is equivalent to the differential inclusion

$$-\dot{z}(t) + RAR^{-1}z(t) + REu(t) \in R^{-1}\partial\psi_{K(t)}(R^{-1}z(t)). \tag{13}$$

Considering the closed convex polyhedral set

$$S(t) := R(K(t)) = \{Rx \mid \; x \in K(t)\}, \tag{14}$$

it is easy to see that $\psi_{S(t)}(x) = (\psi_{K(t)} \circ R^{-1})(x)$ for all $x \in \mathbb{R}^n$. Since R is invertible and symmetric, by Proposition A.2 in the Appendix again, we have

$$\partial\psi_{S(t)}(x) = R^{-1}(\partial\psi_{K(t)})(R^{-1}x) \;\text{ for all } x \in R^n$$

and hence, since $N(S(t); x) = \partial\psi_{S(t)}(x)$, where the normal cone is as in (2), the differential inclusion (13) may be written in the form

$$-\dot{z}(t) + RAR^{-1}z(t) + REu(t) \in N(S(t); z(t)), \tag{15}$$

which appears as the perturbation of a sweeping process. It is clear from the definition of the normal cone in (2) that inclusion (15) is in turn equivalent to the evolution variational inequality

$$\langle \dot{z}(t) - RAR^{-1}z(t) - REu(t), v - z(t)\rangle \geq 0, \quad \forall\, v \in S(t),\ z(t) \in S(t).$$

If $G = 0$ then K does not vary with time and [21, Theorem 2.2] applies with $u(\cdot)$ a continuous mapping with locally L_1 derivative. Here we let K hence S be time-varying, which complicates the analysis. Let us reiterate that the developments made in this section are purely algebraic, i.e., we have only used equivalent ways of writing the right-hand-side, using convex analysis and complementarity theory. If the system has no solution, the notion of equivalence between the different dynamics is void. One notes that the relation in (6) allows us to prove that the considered LCS is of the *gradient* type [1], which is the reason why the transformations into a sweeping process can be performed.

Example 3.1. As an illustration let us consider the electrical circuit whose dynamics is in (8). In this case one trivially has $C = 1$, $B = 1$ so that $P = 1$. Setting $S(t) = \{z \in I\!R \mid z - u(t) \geq 0\}$ one obtains the equivalent form of the dynamics as

$$-\dot{x}(t) \in N(S(t), x(t)) \quad \Leftrightarrow \quad \langle \dot{x}(t), v - x(t)\rangle \geq 0, \quad \forall\, v \in S(t),\ x(t) \in S(t). \qquad (16)$$

3.2. Existence and uniqueness of solutions ($D = 0$)

Let $u(\cdot) : [0, +\infty[\to \mathbb{R}^p$ be a mapping from $[0, +\infty[$ to $\mathbb{R}^p$. In the notation used above and below we identify (when there is no ambiguity) a matrix and the linear mapping associated with it with respect to the usual basis of $\mathbb{R}^n$, $\mathbb{R}^m$ etc. So, the range of the matrix C will be denoted by $\mathrm{Rge}\,(C)$. The one-dimensional Lebesgue measure is denoted as λ.

3.2.1. Introductory material

We now proceed to the analysis of the perturbed sweeping process (15). Consider an interval I of $\mathbb{R}$ and a single-valued mapping $z : I \to \mathbb{R}^n$. We recall that the variation of $z(\cdot)$ on I is the supremum of $\sum \|z(t_i) - z(t_{i-1})\|$ over the set of all finite sets of points $t_0 < t_1 < \cdots < t_k$ of I. When this supremum is finite, the mapping $z(\cdot)$ is said to be of *bounded variation* on I. The mapping $z(\cdot)$ is of *locally bounded variation* on I if it is of bounded variation on each compact subinterval of I.

Considering a set-valued mapping $S : I \rightrightarrows \mathbb{R}^n$ and replacing the above expression $\|z(t_i) - z(t_{i-1})\|$ by the Hausdorff distance $\mathrm{haus}\,(S(t_i), S(t_{i-1}))$, one obtains the concept of set-valued mappings with *bounded variation* on I (resp. *locally bounded variation* on I). The Hausdorff distance between two subsets Q_1 and Q_2 in $\mathbb{R}^n$ is given as usual by

$$\mathrm{haus}\,(Q_1, Q_2) := \max\left\{\sup_{x \in Q_1} d(x, Q_2), \sup_{x \in Q_2} d(x, Q_1)\right\},$$

where $d(x, Q) = \inf\{\|x - y\| : y \in Q\}$. Denote by $\mathrm{var}_S(t)$ the variation of $S(\cdot)$ over $[0, t]$. When the variation function $\mathrm{var}_S(\cdot)$ is locally absolutely continuous on $[0, +\infty[$, the set-valued mapping $S(\cdot)$ is said to be *locally absolutely continuous* on $[0, +\infty[$. As usual the local absolute continuity of the function $v(\cdot) := \mathrm{var}_S(\cdot)$ means that for each $T \in [0, +\infty[$ and for any positive number ε there exists some positive number η such that $\sum_{i=1}^k |v(t_i) - v(s_i)| < \varepsilon$ whenever $\sum_{i=1}^k (t_i - s_i) < \eta$ with $s_i < t_i < s_{i+1}$ in $[0, T]$.

Recall that with any mapping $z : I \to \mathbb{R}^n$ of locally bounded variation on a subinterval I of $\mathbb{R}$ is associated a Radon vector measure, the so-called *differential vector measure* dz of $z(\cdot)$ on I. If, in addition, $z(\cdot)$ is right continuous, this vector measure dz satisfies

$$z(t) = z(s) + \int_{]s,t]} dz \quad \text{for all } s, t \in I \text{ with } s \leq t.$$

If a mapping or a set-valued mapping is right continuous and has a bounded variation (resp. locally bounded variation) on I, we shall say for short that it is *rcbv*. (resp. *locally rcbv*).

For an *rcbv* (resp. locally *rcbv*) mapping $z(\cdot)$ and a positive Radon measure ν on I, the derivative (see, e.g. [26])

$$\frac{dz}{d\nu}(t) := \lim_{\varepsilon \downarrow 0} \frac{dz(I(t,\varepsilon))}{\nu(I(t,\varepsilon))}$$

of dz with respect to ν at t exits for ν-almost every $t \in I$, where $I(t, \varepsilon) := I \cap [t - \varepsilon, t + \varepsilon]$ and where the convention $\frac{0}{0} = 0$ is made. Further, putting $I^-(t, \varepsilon) := I \cap [t - \varepsilon, t]$ and $I^+(t, \varepsilon) := I \cap [t, t + \varepsilon]$, by [34] one also has for ν-almost every $t \in I$

$$\frac{dz}{d\nu}(t) = \lim_{\varepsilon \downarrow 0} \frac{dz(I^-(t,\varepsilon))}{\nu(I^-(t,\varepsilon))} = \lim_{\varepsilon \downarrow 0} \frac{dz(I^+(t,\varepsilon))}{\nu(I^+(t,\varepsilon))}, \tag{17}$$

whenever the vector measure dz is absolutely continuous with respect to ν, and under this absolute continuity $\frac{dz}{d\nu}(\cdot)$ is the density of dz relative to ν.

Throughout unless otherwise stated, the set $S(t)$ will be closed for all t.

Locally absolutely continuous solution: When an initial condition $z_0 \in S(0)$ is fixed and the set-valued mapping $S(\cdot)$ is locally absolutely continuous, the concept of solution of (15) is clear in the sense that it is, as usual, a locally absolutely continuous mapping $z(\cdot)$ for which $z(0) = z_0$ and the inclusion (15) holds for all t outside of a Lebesgue null subset of $[0, +\infty[$. Recall that any locally absolutely continuous mapping $z(\cdot)$ with values in $\mathbb{R}^n$ is differentiable Lebesgue almost everywhere and $z(t) - z(s) = \int_s^t \dot{z}(\tau)\, d\lambda(\tau)$.

Locally *rcbv* solution: Suppose now that $S(\cdot)$ is locally *rcbv* and the mapping $f(\cdot, \cdot)$ with $f(t, y) := -RAR^{-1}y - REu(t)$ is not identically null. Throughout, in such a case we shall denote by $\mu := d\,(\mathrm{var}_S)$ the differential measure of $\mathrm{var}_S(\cdot)$. This Radon measure μ is obviously positive since the function $\mathrm{var}_S(\cdot)$ is non decreasing. According to [15] a mapping $z : [0, +\infty[\to \mathbb{R}^n$ is a solution of (15) with z_0 as initial condition if:

(i) $z(\cdot)$ is locally *rcbv* and satisfies $z(0) = z_0$ and $z(t) \in S(t)$ for all $t \in [0, +\infty[$;

(ii) there exists a positive Radon measure ν absolutely continuously equivalent to the measure $\mu + \lambda$ and with respect to which the differential vector measure dz is absolutely continuous with density $\frac{dz}{d\nu} \in L^1_{loc}([0, +\infty[, \nu; R^n)$ and

$$-\frac{dz}{d\nu}(t) - f(t, z(t))\frac{d\lambda}{d\nu}(t) \in N(S(t); z(t)) \quad \nu - \text{a.e. } t \in [0, +\infty[, \tag{18}$$

where $\frac{d\lambda}{d\nu}(\cdot)$ denotes the density relative to ν of the Lebesgue measure λ which is obviously absolutely continuous with respect to the measure ν.

It is observed in [15] that:

$$\begin{cases} \text{any mapping satisfying (i) is a solution} \\ \text{if and only if (ii) holds with the measure } \mu + \lambda \text{ itself in place of } \nu. \end{cases} \tag{19}$$

So, in the bounded variation case we shall follow [15] in writing (15) in the form

$$\begin{cases} -dz \in N(S(t); z(t)) + f(t, z(t))\, d\lambda \\ z(0) = z_0 \in S(0), \end{cases} \tag{20}$$

where $f(t, y) := -RAR^{-1}y - REu(t)$ for all $y \in \mathbb{R}^n$.

It is known (see [15]) that the existence result for such a differential inclusion requires the $\mathrm{L}^1_{\mathrm{loc}}$-property of the mapping $f(\cdot, y)$ for each $y \in \mathbb{R}^n$. So in the remaining of the paper we shall make the assumption

$$u(\cdot) \in \mathrm{L}^1_{\mathrm{loc}}([0, +\infty[, \lambda; \mathbb{R}^p). \tag{21}$$

Below we shall also have to assume that the set-valued mapping $S(\cdot)$ is of locally bounded variation (resp. is locally absolutely continuous). Before proving our theorem on existence and uniqueness in the linear case, let us establish the following proposition which provides some general sufficient conditions ensuring this behavior for the set-valued mapping $S(\cdot)$.

Proposition 3.2. *Assume that the set-valued mapping $K(\cdot)$ is nonempty-valued, i.e. all the sets $S(t)$ are nonempty (which holds in particular whenever the constraint qualification*

$$\mathrm{Rge}\,(C) - \mathbb{R}^m_+ = \mathbb{R}^m \tag{22}$$

is fulfilled[1]). If the component mapping $Gu(\cdot)$ has a local bounded variation (resp. is locally absolutely continuous) on $[0, +\infty[$ (which obviously holds whenever so is the mapping $u(\cdot)$), then the closed convex set-valued mapping $S(\cdot)$ has a local bounded variation (resp. is locally absolutely continuous) too. In the same way, $S(\cdot)$ is right continuous with respect to the Hausdorff distance whenever $Gu(\cdot)$ is right continuous.

Proof. By the nonemptiness of all $K(t)$ as defined above (which is obviously guaranteed by Assumption (22)) and by Lemma A.1 in the Appendix, there exists some constant $\gamma > 0$ (depending only on the matrix C) such that for all $s, t \in [0, +\infty[$

$$\mathrm{haus}(K(t), K(s)) \leq \gamma \|Gu(t) - Gu(s)\|.$$

Since $S(t) := R(K(t))$, one obtains easily that for all $s, t \in [0, +\infty[$

$$\mathrm{haus}(S(t), S(s)) \leq \|R\|\, \mathrm{haus}(K(t), K(s)) \leq \gamma \|R\|\, \|Gu(t) - Gu(s)\|.$$

[1] The equality in (22) means that for all $x \in \mathbb{R}^m$, there exists $y \in \mathrm{Rge}\,(C)$ and $z \in \mathbb{R}^m_+$ such that $z - y = x$. Obviously it holds whenever the linear mapping associated with C is onto, i.e., the matrix C has rank m, but also in many other cases. See Appendix A.7 for an example.

It is not difficult to check that this last inequality and the local bounded variation (resp. local absolute continuity) of the mapping $Gu(\cdot)$ entails the local bounded variation (resp. local absolute continuity) of the set-valued mapping $S(\cdot)$ on the interval $[0, +\infty[$.

Finally, the right continuity property obviously follows from the same last inequality above. $\square$

In the proof of Theorem 3.5 below we shall need the following result of Moreau [33]. In [33] it has been established in the general Hilbert setting.

Proposition 3.3. *Let ν be a positive Radon measure on a bounded closed interval I and $z(\cdot) : I \to \mathbb{R}^n$ be an* rcbv *mapping whose differential measure dz is absolutely continuous with respect to ν. Then, the function $\Phi : I \to \mathbb{R}$ with $\Phi(t) := \|z(t)\|^2$ is an* rcbv *function whose differential measure $d\Phi$ satisfies, in the sense of the ordering of real measures,*

$$d\Phi \leq 2 \left\langle z(\cdot), \frac{dz}{d\nu}(\cdot) \right\rangle d\nu,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product of $\mathbb{R}^n$.

The following Gronwall-like lemma concerning Radon measures (instead of the usual Lebesgue measure) is also available. It is due to Monteiro Marques [28] (see also [27] for its proof).

Lemma 3.4. *Let ν be a positive Radon measure on a bounded closed interval $[T_0, T]$ and let $g(\cdot)$ be a non negative function in $L^1_\nu([T_0, T]; \mathbb{R})$. Assume that, for some fixed real number $\sigma \in [0, 1[$, one has, for ν-a.e. $t \in]T_0, T]$,*

$$0 \leq g(t)\nu(\{t\}) \leq \sigma < 1.$$

Let a non negative function $\varphi(\cdot) \in L^\infty_\nu([T_0, T]; \mathbb{R})$ and let some fixed real number $\alpha \geq 0$ satisfy for all $t \in [T_0, T]$,

$$\varphi(t) \leq \alpha + \int_{]T_0, t]} g(s)\varphi(s)\, d\nu(s).$$

Then for all $t \in [T_0, T]$ one has

$$\varphi(t) \leq \alpha \exp\left\{ \frac{1}{1-\sigma} \int_{]T_0, t]} g(s)\, d\nu(s) \right\}.$$

3.2.2. Main result

We can now prove the following theorem.

Theorem 3.5. *Assume that (21) holds and that the set-valued mapping $S(\cdot) = R(K(\cdot))$ is locally absolutely continuous (resp. locally* rcbv*) with nonempty values. Then the perturbed differential inclusion (15) with initial condition $z(0) = z_0 \in R(K(0))$ has one and only one locally absolutely continuous (resp. locally* rcbv*) solution $z(\cdot)$ on $[0, +\infty[$.*

Proof. As above, put for every $(t, y) \in [0, +\infty[\times\mathbb{R}^n$

$$f(t, y) := -RAR^{-1}y - REu(t).$$

Obviously the mapping $f(\cdot, \cdot)$ is Lebesgue measurable in t and Lipschitz continuous with respect to the second variable y. Further for

$$k(t) := \max\{\|RE(u(t))\|, \|RAR^{-1}\|\}$$

one has the $\mathrm{L}^1_{\mathrm{loc}}$ linear growth condition

$$\|f(t, y)\| \leq k(t)(1 + \|y\|) \text{ for all } (t, z) \in [0, +\infty[\times\mathbb{R}^n. \tag{23}$$

Using Lemma 3.2 and the properties above concerning the mapping $f(\cdot, \cdot)$ we may apply Theorem 1 in [16] (see its statement in Theorem A.4 and related comments in the Appendix A.5) to obtain, in the case where $S(\cdot)$ is nonempty-valued and absolutely continuous, that the perturbed sweeping process differential inclusion

$$\begin{cases} -\dot{z}(t) \in N(S(t); z(t)) + f(t, z(t)) \\ z(0) = z_0 \in S(0) \end{cases}$$

has one and only one locally absolutely continuous solution on $[0, +\infty[$. The above equivalence between (13) and (15) with the initial conditions $z(0) = z_0 \in R(K(0))$ and $z(0) = z_0 \in S(0)$ respectively yields the conclusion of the theorem in the absolutely continuous case.

Suppose now that the set-valued mapping $S(\cdot)$ is nonempty-valued and locally *rcbv* on $[0, +\infty[$. Theorem 3.1 in [15] (see its statement in Theorem A.5 and related comments in the Appendix A.6) ensures that differential inclusion (20) has at least a solution $z(\cdot)$ which is locally *rcbv*. Let us prove the uniqueness. Consider two locally *rcbv* solutions $z_1(\cdot)$ and $z_2(\cdot)$ and fix any number $T > 0$. Let μ be the Radon measure associated with the variation function $\mathrm{var}_S(\cdot)$ and put as above $\nu := \mu + \lambda$. By (19) with respect to this positive Radon measure ν the differential measures dz_i and the Lebesgue measure λ are absolutely continuous, for $i = 1, 2$, and

$$-\frac{dz_i}{d\nu}(t) - f(t, z_i(t))\frac{d\lambda}{d\nu}(t) \in N(S(t); z_i(t)) \quad \nu - a.e.\ t \in [0, T].$$

Then we have for ν-almost all $t \in [0, T]$

$$\left\langle \frac{dz_1}{d\nu}(t) + f(t, z_1(t))\frac{d\lambda}{d\nu}(t) - \frac{dz_2}{d\nu}(t) - f(t, z_2(t))\frac{d\lambda}{d\nu}(t), z_1(t) - z_2(t) \right\rangle \leq 0$$

hence for $\beta := \|RAR^{-1}\|$

$$\left\langle \frac{dz_1}{d\nu}(t) - \frac{dz_2}{d\nu}(t), z_1(t) - z_2(t) \right\rangle \leq \beta\|z_1(t) - z_2(t)\|^2 \frac{d\lambda}{d\nu}(t).$$

According to Proposition 3.3, for any $t \in [0, T]$ we have

$$\|z_1(t) - z_2(t)\|^2 \leq \int_{]0,t]} 2\beta\|z_1(s) - z_2(s)\|^2 \frac{d\lambda}{d\nu}(s)\, d\nu(s),$$

i.e.,

$$\|z_1(t) - z_2(t)\|^2 \leq \int_{]0,t]} 2\beta\|z_1(s) - z_2(s)\|^2 \, d\lambda(s).$$

The usual Gronwall Lemma (see, e.g., [13, Proposition VI-9]) yields $z_1(t) = z_2(t)$ for all $t \in [0, T]$ and hence the uniqueness property is established and the proof is complete. □

Remark 3.6. The terms $Eu(t)$ and $Gu(t)$ in (5) play a different role both in the conversion from complementarity system to a sweeping process, and in the well-posedness proof, see (13) and (12), in the sense that one is not obliged to impose the same properties on both terms to get the well-posedness result. Indeed according to [16] and the proof of Theorem 3.5 it is sufficient to assume that the component mapping $Eu(\cdot)$ (instead of the mapping $u(\cdot)$ itself) is an $\mathrm{L}^1_{\mathrm{loc}}$ mapping. This assumption has the advantage to envisage in the proofs of Lemma 3.2 and Theorem 3.5, $Eu(t)$ and $Gu(t)$ may not involve the same components of $u(t)$.

Remark 3.7. When the solution is *rcbv*, it may possess jumps. The jumps may be deduced from (20) by noting that state jumps correspond to atoms of the measure dz, so that (20) may be rewritten at such atoms as

$$-z(t^+) + z(t^-) \in N(S(t^+); z(t^+)), \tag{24}$$

that is equivalent, provided $S(t)$ is a nonempty convex set, to

$$z(t^+) = \mathrm{prox}\,[S(t^+); z(t^-)], \tag{25}$$

which in turn is equivalent to

$$z(t^+) = \mathrm{argmin}_{z \in S(t^+)} \frac{1}{2}||z - z(t^-)||^2, \tag{26}$$

i.e., $z(t^+)$ ($= z(t)$) is the (unique) closest vector to $z(t^-)$ inside $S(t^+)$ (equivalently, the projection of $z(t^-)$ on $S(t^+)$ in the Euclidean metric). If $z_0 \notin R(K(0))$ then an initial jump has to be imposed on $z(0)$. Then the above result holds on $]0, +\infty[$. In [10] the well-posedness of linear complementarity systems as in (5) has been shown, when the quadruplet (A, B, C, D) is positive real, and the mapping $u(\cdot)$ is piecewise Bohl (Bohl functions are continuous functions possessing a rational Laplace transform). Specifically, [10, Theorem 7.5] proves the global existence and uniqueness of a solution with $x(\cdot)$ in $L_2(\mathbb{R}^+)$, and $\zeta(\cdot)$ is a measure whose singular part has a support contained in the set of discontinuity times of $Gu(t)$ union the initial time. Notice that the set of locally *rcbv* functions contains piecewise Bohl functions. Thus we consider more general inputs $u(\cdot)$ than [10, Theorem 7.5]. However since a function may belong to $L_2(\mathbb{R}^+)$ and not be locally *rcbv*, and vice-versa, we conclude that our results and those of [10, Theorem 7.5] are different. As shown in Section 4 our framework extends to a class of nonlinear complementarity systems. Finally [10] admits general matrices $D \geq 0$, but restricts the analysis to positive real systems [9] with B full-column rank, which is not our case.

3.3. The case $D \geq 0$

As pointed out in the introduction, it is worth considering such cases, because many systems (like electrical circuits) do not have a zero feedthrough matrix D, but a semi positive definite D. In this section our objective is just to point out that extensions of the above developments led for $D = 0$ are possible when $D \neq 0$. We first restrict ourselves to semi positive definite matrices D of the form

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{27}$$

where $D_1 > 0$ is square of dimension $q < m$, and not necessarily symmetric. Actually it is sufficient to suppose that D_1 is a P-matrix for the analysis of this section to work. In such a case D is a particular type of P_0-matrix. Let us partition the multiplier as $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$, such that $D\zeta = \begin{pmatrix} D_1\zeta_1 \\ 0 \end{pmatrix}$. The rest of the vector w is partitioned similarly. The complementarity conditions in (5) can therefore be written as:

$$\begin{cases} 0 \leq \zeta_1(t) \perp w_1(t) = C_1 x(t) + D_1\zeta_1(t) + G_1 u(t) + F_1 \geq 0 \\ 0 \leq \zeta_2(t) \perp w_2(t) = C_2 x(t) + G_2 u(t) + F_2 \geq 0. \end{cases} \tag{28}$$

The first set of conditions is a linear complementarity problem with unknown ζ_1 and matrix D_1. It possesses a unique solution that is continuous piecewise linear in $x(t)$ and $u(t)$ (see Section A.4 in the Appendix), and which we denote as $\zeta_1(x, u)$. Consequently the LCS in (5) is rewritten as:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1\zeta_1(x(t), u(t)) + B_2\zeta_2(t) + Eu(t) \\ 0 \leq \zeta_2(t) \perp w_2(t) = C_2 x(t) + G_2 u(t) + F_2 \geq 0 \end{cases} \tag{29}$$

which we may name a *piecewise linear* CS (PLCS). As such it is also a nonlinear CS. If we suppose that Assumption 1.1 holds for the pair (B_2, C_2), i.e., $P_2 B_2 = C_2^T$ for some $P_2 = P_2^T > 0$, and R_2 the symmetric positive definite square root of P_2, then following the same steps as in Section 3.1 one may rewrite (29) as the inclusion

$$-\dot{z}(t) + R_2(AR_2^{-1}z(t) + B_1\zeta_1(R_2^{-1}z(t), u(t))) + R_2 Eu(t) \in N(S(t); z(t)) \tag{30}$$

with $S(t) = R_2(K(t)) = \{R_2 x : x \in K(t)\}$ and $K(t) = \{x \in \mathbb{R}^n : C_2 x + G_2 u + F_2 \geq 0\}$. Then Theorem 3.5 may be applied since the steps of its proof can be redone for a Lipschitz continuous vector field. One may also apply the results of the nonlinear case of Section 4.

Let us now outline a way to transform the LCS in (5) so that the new feedthrough matrix has the block-diagonal form in (27). Denoting as $\mathbb{R}^{m\times n}$ the space of real matrices with m rows and n columns, the following holds.

Lemma 3.8. *Let $D \in I\!R^{m\times m}$ have rank $q < m$, and suppose that there exist full-rank matrices V and W with $W^T = V$, such that $D = V\bar{D}W$ with $\bar{D} = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$, where D_1 is a $q \times q$ full rank matrix. Then the LCS in* (5) *can be equivalently rewritten as the cone complementarity system* (CCS)

$$\begin{cases} \dot{x}(t) = Ax(t) + Eu(t) + BW^{-1}\bar{\zeta} \\ C_{W^{-1}} \ni \bar{\zeta} \perp \bar{w} = V^{-1}Cx(t) + V^{-1}Gu(t) + V^{-1}F + \bar{D}\bar{\zeta} \in C_V \end{cases} \tag{31}$$

where $C_V = \{z \in I\!R^m \ : \ Vz \in I\!R^m_+\} := V^{-1}(I\!R^m_+)$, and $C_{W^{-1}} = \{z \in I\!R^m \ : \ W^{-1}z \in I\!R^m_+\} := W(I\!R^m_+)$ are two dual polyhedral cones.

The second line in (31) is a cone complementarity problem (CCP) [18, Definition 1.1.2]. We sketch the proof of the lemma. It consists of using the complementarity variables $\bar{\zeta} = W\zeta$ and $\bar{w} = V^{-1}w$ and of noting that if $(\bar{\zeta}, \bar{w})$ is a solution of the CCP in (31), then (ζ, w) is a solution of the CP in (5), and vice-versa. Moreover $B\zeta = BW^{-1}\bar{\zeta}$, so the right-hand-sides of the LCS and of the CCS are the same. Thus if both systems are well-posed their solutions coincide for each admissible initial state $x(0)$. Notice that the condition $W^T = V$ is due to the orthogonality which has to be preserved between the two complementarity variables $\bar{\zeta}$ and $\bar{w}$. The duality of the two cones follows by direct calculation and using $W^T = V$: let $z \in C_{W^{-1}}$ and $y \in C_V$, then $\langle z, y\rangle = \langle Wz', V^{-1}y'\rangle = \langle z', y'\rangle \geq 0$ for some z' and $y' \in I\!R^m_+$. It is noteworthy that the CCS in (31) represents a special case of the LCS in (5) with a feedthrough matrix as in (27), where the linear complementarity problem is replaced by a CCP. The basic idea is then to transform the CCS (31) into a sweeping process, following the same lines as in Section 3.1 and in (28) through (30), since a CCP with two dual cones can be equivalently written as an inclusion into a normal cone, see [18, Chapter 1]. Further conditions have to be imposed on D_1 to assure that these manipulations are doable. For the sake of briefness the case $D \neq 0$ is not tackled further in this paper.

3.4. Comments on the complementarity conditions

The foregoing subsections are devoted to show the well-posedness of the differential inclusion in (15), which is written as an inclusion of measures in (20) (or an inclusion of densities in (18)) when solutions are locally *rcbv*. The passage from the complementarity system in (5) to the complementarity system in (10) is done thanks to the state variable change $z = Rx$. The passage from the complementarity system in (10) to the differential inclusion is done thanks to the equivalence

$$0 \leq \zeta(t) \perp w(t) \geq 0 \Leftrightarrow -\zeta(t) \in \partial\psi_Q(w(t)), \quad Q = \mathbb{R}^m_+,$$

which holds for vectors $w(t), \zeta(t) \in \mathbb{R}^m$. Let us now make the following observation: when $u(\cdot)$ is locally *rcbv*, then $z(\cdot)$ is locally *rcbv* and may possess jumps. Therefore ζ is a measure, whose atoms coincide with the times of jumps of $z(\cdot)$ and of $u(\cdot)$. The complementarity relation in (10) becomes meaningless at such atoms since it involves the product of a measure with a time-discontinuous function.

Such a problem is troublesome as it means that the product $\zeta^T w$ is not defined as a product of Schwarz' distributions at the atoms of dz (this difficulty is already present in the setting of nonsmooth mechanical systems, see e.g. [6, §1.2.2]). In dissipative systems theory, this product is called the supply rate and has a clear energetical meaning [9]. It is possible to give a mathematical meaning to the input-output product $\langle \zeta, w(t)\rangle = \int \zeta^T w$ by constructing a measure from a functional. More precisely let $\zeta = \delta_t$, the Dirac measure at time t, and let $\varphi(\cdot)$ be right-continuous at t. The space of functions which are δ_t-integrable contains functions continuous at t, and also all the functions $\varphi(\cdot)$ which are δ_t-almost everywhere equal to an integrable (continuous) function $g(\cdot)$.

Since the support of δ_t is $\{t\}$, it is sufficient that $\varphi(t) = g(t)$. Then

$$\langle \varphi, \delta_t \rangle = \int \varphi d\delta_t = \int g d\delta_t = \langle g, \delta_t \rangle = g(t) = \varphi(t) = \varphi(t^+).$$

This may be a path to properly define the complementary-slackness variables product over any time interval $[0, \tau]$ with $\tau > 0$. However as explained below the framework of this paper allows us to solve this issue without going into such abstract measures considerations.

In this paper we started from a complementarity formulation (5) and then constructed a differential inclusion which is given under its more general form in (20). However in view of the above observation, it may be more logical to interpret our result in the reverse sense: inclusion (20) or equivalently (18) is given in the case of locally absolutely continuous solutions by inclusion (15). Given the definition of the set $S(t)$ in (14), the variable change $z = Rx$ allows one to conclude that provided $u(\cdot)$ is locally absolutely continuous, then (15) is equivalent to (10) which in turn is equivalent to (5) (equivalence means that if $z(\cdot)$ is the unique solution of (15) with initial data z_0, then $x(\cdot) = R^{-1}z(\cdot)$ is the unique solution of (10) with initial data $x_0 = R^{-1}z_0$). In the case $z(\cdot)$ is locally *rcbv*, the measure differential inclusion in (20) appears to be a more general formalism than (5) which *per se* cannot handle state jumps. The measure differential inclusion in (20) allows us to derive a state jump as shown in (24) (25), and to give a meaning to the dynamics at the atoms of dz. Moreover as exposed in the next section the differential inclusion formalism, that is originally constructed for nonlinear time-varying perturbations $f(t, z)$, is quite useful for the study of a class of nonlinear non-autonomous complementarity systems.

Let us come back on the issue raised above concerning the product $\zeta^T w$. At atoms of dz one has $\frac{dz}{d\nu}(t) = \beta(z(t^+) - z(t^-))$ for some $\beta > 0$, while $\frac{d\lambda}{d\nu}(t) = 0$ because the Lebesgue measure λ has no atom. Then (18) is equivalent to (24) or (25). In other words, there is a $\bar{\zeta}(t) \in -N(S(t); z(t^+))$ such that $z(t^+) - z(t^-) = \bar{\zeta}(t)$. The function $\bar{\zeta}(t)$ is the density of ζ at the atom t with respect to $d\nu$, i.e., the magnitude of the Dirac measure ζ, and we may write it as $\bar{\zeta}(t) = \frac{d\zeta}{d\nu}(t)$. We may consequently write the input-output product associated with the differential inclusion (18) as

$$\begin{aligned}\left\langle \frac{d\zeta}{d\nu}(t), w(t) \right\rangle &= \left\langle \frac{d\zeta}{d\nu}(t), \frac{1}{2}CR^{-1}(z(t^+) + z(t^-)) + Gu(t)\frac{d\lambda}{d\nu}(t) + F\frac{d\lambda}{d\nu}(t) \right\rangle \\ &= \left\langle \frac{d\zeta}{d\nu}(t), \frac{1}{2}CR^{-1}(z(t^+) + z(t^-)) \right\rangle.\end{aligned} \tag{32}$$

Let us place ourselves in the perspective of positive real systems [9], that satisfy (6) for some matrix P. Let $V(\cdot)$ be a storage function for the triple (A, B, C). The infinitesimal dissipation equality $\frac{dV}{d\nu} = \langle \zeta, w(t) \rangle$ is equal at the atoms t of dz to

$$V(t^+) - V(t^-) = \bar{\zeta}^T w(t) = \frac{1}{2}\bar{\zeta}^T CR^{-1}(z(t^+) - z(t^-)).$$

Using $C = B^T P = B^T R^2$ and the algebraic form of the dynamics at atoms of dz (i.e. $z(t^+) - z(t^-) = RB\bar{\zeta}(t)$), this can be rewritten as

$$V(t^+) - V(t^-) = \frac{1}{2}z^T(t^+)z(t^+) - \frac{1}{2}z^T(t^-)z(t^-) \tag{33}$$

where one recalls that the storage functions are simply $V(z) = z^T z$ in the $z-$coordinates. From (25) one deduces that $V(\cdot)$ enjoys non positive jumps. The equality in (33) is interesting because of its energetical interpretation (storage functions may be thought of as energy functions). A similar development is proposed in [5, 9] for Lagrangian nonsmooth systems embedded in the so-called Moreau's second order sweeping process. The product in (32) obviously is zero outside the atoms of dz, because of the complementarity condition. However the system "dissipates" at the state jumps.

4. Nonlinear complementarity systems

4.1. Dynamics and basic assumptions

We now focus our attention on nonlinear complementarity systems of the form

$$\begin{cases} \dot{x}(t) = a(x(t)) + B\zeta(t) + e(x(t), u(t)) \\ 0 \leq \zeta(t) \perp w(t) = c(x(t)) + g(u(t)) \geq 0, \end{cases} \tag{34}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^m$, $B \in \mathbb{R}^{n\times m}$, the mappings $a(\cdot)$ and $e(\cdot)$ are continuous and $g(\cdot)$ is supposed to be locally Lipschitz continuous, and the regularity of $c(\cdot)$ and $u(\cdot)$ will be specified later. The class of complementarity systems in (34) encapsulates (5), so that the material in this section is a generalization of the linear case of Section 3. We shall proceed similarly as in Section 3, i.e., we shall first show how one may transform (34) into a perturbed sweeping process, then some well-posedness results will be shown. One major discrepancy with the linear complementarity case, is that the sets $S(t)$ will no longer necessarily be convex (such an assumption would be much too stringent in the nonlinear case). Instead, convexity will be replaced by the more general notion of r-prox-regularity. Roughly speaking, r-prox-regular sets have the property that the projection of a close enough point is unique. A definition is given in the Appendix. Let us now make the following basic assumptions on the system (34), which generalize Assumption 1.1.

Assumption 4.1. The system (34) has the following "input-output" property: there exists a function $V(\cdot)$ such that

(i)

$$c^T(x) = \frac{\partial V}{\partial x}^T(x)B. \tag{35}$$

(ii) $V(\cdot)$ is of class $C^3(\mathbb{R}^n; \mathbb{R}_+)$.

(iii) The Hessian $\frac{\partial^2 V}{\partial x^2}(x)$ is positive definite and symmetric for all $x \in \mathbb{R}^n$.

The reason why we ask for C^3 property and not C^2 will appear later.

4.2. State space transformation

Let us perform the state transformation $z = h(x)$, with

$$\frac{\partial h}{\partial x}^T(x) = \left(\frac{\partial^2 V}{\partial x^2}(x)\right)^{\frac{1}{2}} =: \Lambda(x). \tag{36}$$

(Note that $\Lambda(x)$ inherits the symmetric property of $\frac{\partial^2 V}{\partial x^2}(x)$). We therefore implicitly assume that $\left(\frac{\partial^2 V}{\partial x^2}(x)\right)^{\frac{1}{2}}$ is integrable, and that $h(\cdot)$ is a diffeomorphism from $\mathbb{R}^n$ into $\mathbb{R}^n$. We obtain

$$\begin{cases} \dot{z}(t) = \frac{\partial h}{\partial x}^T(x)a(h^{-1}(z(t))) + \frac{\partial h}{\partial x}^T(x)B\zeta(t) + \frac{\partial h}{\partial x}^T(x)e(h^{-1}(z(t)), u(t)) \\ 0 \leq \zeta(t) \perp w(t) = c(h^{-1}(z(t))) + g(u(t)) \geq 0. \end{cases} \tag{37}$$

Using the equivalence $(0 \leq \zeta(t) \perp w(t) \geq 0) \Leftrightarrow -\zeta(t) \in \partial\psi_{\mathbb{R}^m_+}(w(t))$, one may rewrite (37) as the inclusion

$$\begin{aligned} &-\dot{z}(t) + \frac{\partial h}{\partial x}^T(x)a(h^{-1}(z(t))) + \frac{\partial h}{\partial x}^T(x)e(h^{-1}(z(t)))u(t) \\ &\quad \in \frac{\partial h}{\partial x}^T(x)B\, \partial\psi_{\mathbb{R}^m_+}(c(h^{-1}(z(t))) + g(u(t))). \end{aligned}$$

Using (35) and (36) we have that $\Lambda(x)B = (\Lambda(x))^{-1}(\Lambda(x))^2 B$ and $\left(\frac{\partial c}{\partial x}\right)^T = (\Lambda(x))^2 B$, so we get

$$\begin{aligned} &-\dot{z}(t) + \frac{\partial h}{\partial x}^T(x)a(h^{-1}(z(t))) + \frac{\partial h}{\partial x}^T(x)e(h^{-1}(z(t)), u(t)) \\ &\quad \in (\Lambda(x))^{-1}\left(\frac{\partial c}{\partial x}\right)^T \partial\psi_{\mathbb{R}^m_+}(c(h^{-1}(z(t))) + g(u(t))). \end{aligned}$$

Setting $S(t) := \{z \in I\!R^n \mid c(h^{-1}(z)) + g(u(t)) \geq 0\}$ and $\Phi_t(z) := c \circ h^{-1}(z) + g(u(t))$, we see that $\psi_{S(t)} = \psi_{\mathbb{R}^m_+} \circ \Phi_t$.

Taking into account condition (59) in the Appenedix for the chain rule and the fact that from (36) one has $\nabla\Phi_t(z) = B^T\Lambda(h^{-1}(z))$, the last equality concerning $\psi_{S(t)}$ leads us to assume that there exists some constant $\rho > 0$ such that for all $x \in \mathbb{R}^n$

$$\rho\mathbb{B}_{\mathbb{R}^m} \subset B^T\Lambda(x)(\mathbb{B}_{\mathbb{R}^n}) - \mathbb{R}^m_+, \tag{38}$$

where $\mathbb{B}_{\mathbb{R}^m}$ denotes the Euclidean closed unit ball in $\mathbb{R}^m$ centered at the origin. The inclusion in (38) parallels (22), and in particular secures that the image of the mapping $B^T\Lambda(\cdot)$ has a positive measure in $\mathbb{R}^n$. Note that (38) and Assumption 2 assure us that the set $S(t)$ is normally regular (see Proposition A.3 in the Appendix).

It is worth noting that $S(t)$ may not be a convex set, so that the notions of subdifferential and normal cone of convex analysis used in the previous section have to be extended to more general (nonconvex) objects. Since $\nabla\Phi_t(z) = B^T\Lambda(h^{-1}(z))$, Assumption (38) along with Assumption 4.1(ii) obviously ensure that $\mathbb{R}^m = \mathrm{Rge}\,\nabla\Phi_t(z) - \mathbb{R}^m_+$, and hence $\mathbb{R}^m = \mathrm{Rge}\,\nabla\Phi_t(z) - (\mathbb{R}^m_+ - \Phi_t(z))$. So, for the convex function $\psi_{\mathbb{R}^m_+}(\cdot)$ and the Jacobian at $z(t)$ of $\Phi_t(\cdot)$, the equality $\mathbb{R}^m = \mathrm{Rge}\,\nabla\Phi_t(z(t)) - \mathbb{R}_+(R^m_+ - \Phi_t(z(t)))$ holds, that is, the requirement (59) of Proposition A.3 in the Appendix is fullfilled for the functions involved in $\psi_{S(t)} = \psi_{\mathbb{R}^m_+} \circ \Phi_t$. Therefore the equality $\frac{\partial c \circ h^{-1}}{\partial z} = \frac{\partial c}{\partial x}\left(\frac{\partial h}{\partial x}\right)^{-1}$ allows us to translate the last differential inclusion into

$$-\dot{z}(t) + \Lambda(x)a(h^{-1}(z(t))) + \Lambda(x)e(h^{-1}(z(t)), u(t)) \in \partial\psi_{S(t)}(z(t)), \tag{39}$$

where $\partial\psi_{S(t)}(z(t))$ denotes (see [38, 29]) the (Mordukhovich) basic subdifferential of the function $\psi_{S(t)}$. Recalling that the basic subdifferential of the indicator function $\psi_Q(\cdot)$ of a set Q is nothing else but its basic normal cone $N(Q;\cdot)$, (see Section A.1 for the definitions of basic subdifferential and basic normal cone), we may rewrite the inclusion in (39) more compactly as follows:

$$-\dot{z}(t) + \tilde{h}(z(t)) + \tilde{e}(z(t), u(t)) \in N(S(t); z(t)). \tag{40}$$

To conclude this subsection, we point out that Assumption 2 and (38) allow us to tranform (34) into (40), relying on nonsmooth analysis tools. Assuming the convexity of all the sets and functions in such a nonlinear framework would be far too much restrictive.

4.3. Existence and uniqueness of solutions

As a first step in the arguments of our main well-posedness results, we establish the following lemma. In its proof we use the main idea of the development of the sufficiency part of Theorem 9.40 in [38] but we deal with the inequality below in all the space $\mathbb{R}^n$. (Of course, the inequality can be seen as a *global* metric regularity). We recall that the distance from a point y to a subset Q of $\mathbb{R}^n$ is defined as $d(y,Q) = \inf_{u\in Q} \|y-u\|$ and as usual we adopt the convention that $d(y,Q) = +\infty$ when the set Q is empty.

Lemma 4.2. *Assume that for the mapping $k : \mathbb{R}^n \to \mathbb{R}^m$ with $k(y) := c \circ h^{-1}(y)$ there exists some $v_0 \in \mathbb{R}^m$ such that $k^{-1}(\mathbb{R}^m_+ - v_0) \neq \emptyset$. Then under the assumption in* (38) *and for any fixed $\hat{y} \in \mathbb{R}^n$, the distance function $v \mapsto d(\hat{y}, k^{-1}(\mathbb{R}^m_+ - v))$ is finite and Lipschitz continuous with $\frac{1}{\rho}$ as Lipschitz constant on all $\mathbb{R}^n$ (and hence in particular $k^{-1}(\mathbb{R}^m_+ - v)) \neq \emptyset$ for all $v \in \mathbb{R}^m$).*

Proof. Considering the set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $M(y) := -k(y) + \mathbb{R}^m_+$, we have

$$d(\hat{y}, M^{-1}(v)) = d(\hat{y}, k^{-1}(\mathbb{R}^m_+ - v)) =: \varphi(v) \in \mathbb{R}_+ \cup \{+\infty\}.$$

Fix any real number α and take any sequence $(v_i)_{i\in\mathbb{N}}$ converging to v and satisfying $\varphi(v_i) \le \alpha$. Choose $y_i \in M^{-1}(v_i)$ with $\varphi(v_i) = \|\hat{y} - y_i\|$. The sequence $(y_i)_i$ is bounded and hence without loss of generality we may suppose that it converges to some y in $\mathbb{R}^n$. Obviously $\|\hat{y} - y\| \le \alpha$ and it is easy to see that $y \in M^{-1}(v)$. Therefore, $\varphi(v) \le \alpha$ and this yields that $\varphi(\cdot)$ is lower semicontinuous. Further, we observe that $\varphi(\cdot)$ is proper because $\varphi(v_0) < +\infty$ according to the nonemptiness assumption of $k^{-1}(\mathbb{R}^m_+ - v_0)$.

Fix now any $(\bar{v}, v^*)$ in the graph of the Fréchet subdifferential of $\varphi(\cdot)$ (see Section A.3 for the definition) and choose $\bar{y} \in M^{-1}(\bar{v})$. Then for each $\varepsilon > 0$ there exists some neighborhood W of $\bar{v}$ such that for all $v \in W$

$$\langle v^*, v - \bar{v}\rangle \le \varphi(v) - \varphi(\bar{v}) + \varepsilon\|v - \bar{v}\|,$$

which means for all $y \in \mathbb{R}^n$

$$\begin{aligned}&\langle 0, y - \bar{y}\rangle + \langle v^*, v - \bar{v}\rangle \\ &\le \|y - \hat{y}\| + \psi_{\operatorname{gph} M}(y, v) - \|\bar{y} - \hat{y}\| - \psi_{\operatorname{gph} M}(\bar{y}, \bar{v}) + \varepsilon\|y - \bar{y}\|,\end{aligned}$$

where $\operatorname{gph} M := \{(y, v) \mid v \in M(y)\}$ denotes the graph of the set-valued mapping $M(\cdot)$ and $\psi_{\operatorname{gph} M}(\cdot)$ is the indicator function of the graph of $M(\cdot)$. Thus, $(0, v^*)$ is a

Fréchet subgradient of the function $\sigma(y,v) := \|y-\hat{y}\| + \psi_{\operatorname{gph} M}(y,v)$ at $(\bar{y},\bar{v})$. Further, putting $K(y,v) := v + k(y)$ and observing that $\psi_{\operatorname{gph} M} = \psi_{\mathbb{R}^m_+} \circ K$, it is not difficult to see through Section A.3 in the Appendix that the function $\psi_{\operatorname{gph} M}$ is subdifferentially regular. The two functions in the definition of $\sigma(\cdot)$ being subdifferentially regular and the first one being convex continuous (hence locally Lipschitz), the subdifferential sum rule provides some $y^* \in \mathbb{B}_{\mathbb{R}^n}$ such that (y^*, v^*) is a Fréchet normal to $\operatorname{gph} M$ at $(\bar{y},\bar{v})$. Putting $\bar{p} = \bar{v} + k(\bar{y}) \in \mathbb{R}^m_+$, it is not difficult to translate the latter into $v^* \in N(R^m_+;\bar{p})$ and $y^* = v^* \circ \nabla k(\bar{y})$. Note also that $\nabla k(y) = B^T \Lambda(h^{-1}(y))$ for all y.

Using the inclusion $y^* \in \mathbb{B}_{\mathbb{R}^n}$ and the inclusion in (38), we obtain for any $q \in \mathbb{B}_{\mathbb{R}^m}$ some $q' \in \mathbb{B}_{\mathbb{R}^n}$ and $p \in \mathbb{R}^m_+$ such that

$$\rho\langle v^*, q\rangle = \langle v^*, \nabla k(\bar{y})(q') + p\rangle \le \langle v^* \circ \nabla k(\bar{y}), q'\rangle = \langle y^*, q'\rangle \le 1,$$

which entails $\|v^*\| \le 1/\rho$. According to this Fréchet subdifferential boundedness property and to the lower semicontinuity and properness of $\varphi(\cdot)$, the conclusion of the lemma follows from [41, Theorem 2.1], for example. $\square$

4.3.1. The locally absolutely continuous case (no state jumps)

With Lemma 4.2 at hand, we prove our theorem relative to nonlinear complementarity systems, where we notice that the regularity of the function $c(\cdot)$ is secured by (35). In the statement of the theorem, $\frac{\partial^2 k}{\partial y^2}(y)$ denotes the second derivative at y of the vector-valued mapping $k : \mathbb{R}^n \to \mathbb{R}^m$, as a vector-valued bilinear mapping, and $\|\frac{\partial^2 k}{\partial y^2}(y)\|$ is as usual the standard norm of this bilinear mapping.

Theorem 4.3. *Consider the system in* (34) *and suppose that* $a(\cdot)$, $e(\cdot,\cdot)$ *are continuous and* $g(\cdot)$ *is locally Lipschitz continuous. Suppose also that Assumption* 4.1 *and inclusion* (38) *hold. Let* $u(\cdot)$ *be locally absolutely continuous, and* $z_0 \in S(0)$. *Then the following assertions hold.*

- (a) *There exists some* $T > 0$ *such that the perturbed differential inclusion* (40) *with* z_0 *as initial condition has at least one absolutely continuous solution on* $[0,T]$.
- (b) *For* $k := c \circ h^{-1}$ *as in Lemma* 4.2 *and* $r = \rho\big(\sup_{y\in \operatorname{co}(\operatorname{Rge} S)} \|\frac{\partial^2 k}{\partial y^2}(y)\|\big)^{-1}$, *all the sets* $S(t)$ *are* r*-prox-regular, and the solution is unique, whenever* $\frac{\partial^2 k}{\partial y^2}(\cdot)$ *is bounded on the convex hull* $\operatorname{co}(\operatorname{Rge} S)$ *of* $\operatorname{Rge} S$.
- (c) *If in addition,* $a(\cdot)$ *and* $e(\cdot, u(t))$ *are Lipschitz continuous on bounded subsets of* $\mathbb{R}^n$, *where the dependence in* t *of the Lipschitz constant of the mapping* $e(\cdot,u(t))$ *is* $\mathrm{L}^1_{\mathrm{loc}}([0,+\infty[;\mathbb{R})$, *and if the mapping* $(t,y) \mapsto \tilde{h}(y) + \tilde{e}(y,u(t))$ *in* (40) *satisfies an* $\mathrm{L}^1_{\mathrm{loc}}$ *linear growth condition like* (23), *then one has a unique locally absolutely continuous solution on all the interval* $[0,+\infty[$.

Proof. First, observe that, according to Lemma 4.2 and to the inclusion $z_0 \in S(0)$, all the sets $S(t)$ are nonempty. Further, as already noticed, the sets $S(t)$ are normally regular. Note also that the statement in Lemma 4.2 translates that the inverse $M^{-1}(\cdot)$ of the set-valued mapping $M(\cdot)$ with $M(y) = -k(y) + \mathbb{R}^m_+$ is Lipschitz continuous on $\mathbb{R}^m$ with respect to the Hausdorff distance (with $\frac{1}{\rho}$ as Lipschitz constant). Since $S(t) = M^{-1}(g(u(t))$, the above assumptions easily give the local absolute continuity of the set-valued mapping $S(\cdot)$ with respect to the Hausdorff distance. Put $f(t,y) :=$

$-\tilde{h}(y) - \tilde{e}(y, u(t))$ for all $y \in \mathbb{R}^n$, with $\tilde{h}(\cdot)$ and $\tilde{e}(\cdot)$ given by the translation of (39) into (40). Fix any positive number $\tau > \|z_0\|$ and take

$$\tilde{f}(t, y) := \begin{cases} f(t, y) & \text{if } \|y\| \le \tau \\ f\left(t, \dfrac{\tau}{\|y\|} y\right) & \text{if } \|y\| \ge \tau. \end{cases} \tag{41}$$

This mapping $\tilde{f}(\cdot, \cdot)$ is continuous and bounded on $[0, \tau] \times \mathbb{R}^n$. From [40, Theorem 4.4] the differential inclusion

$$-\dot{z}(t) \in N(S(t); z(t)) + \tilde{f}(t, z(t)) \tag{42}$$

with initial condition $z_0 \in S(0)$ has at least one absolutely continuous solution $z(\cdot)$ on $[0, \tau]$. Since $z(\cdot)$ is continuous and $\|z_0\| < \tau$, we may choose some positive number $T \le \tau$ such that $\|z(t)\| \le \tau$ for all $t \in [0, T]$. Then for all $t \in [0, T]$ we have $\tilde{f}(t, z(t)) = f(t, z(t))$ and hence from (42) we obtain that $z(\cdot)$ is a solution of (40) on $[0, T]$, which proves (a).

Suppose now the boundedness of $\frac{\partial^2 k}{\partial y^2}(\cdot)$ on $\operatorname{co}(\operatorname{Rge} S)$ and fix any $t \ge 0$. Take $y_i^* \in N(S(t); y_i)$ for $i = 1, 2$. Since $\psi_{S(t)} = \psi_{\mathbb{R}^m_+} \circ \Phi_t$, where $\Phi_t(y) := k(y) + g(u(t))$ with $k(\cdot)$ as in Lemma 4.2, the convexity of $\psi_{\mathbb{R}^m_+}(\cdot)$ and (38) allow us, through the same arguments yielding to (39), to apply Proposition A.3 to obtain some $v_i^* \in N(\mathbb{R}^m_+; \Phi_t(y_i))$ such that $y_i^* = v_i^* \circ \nabla k(y_i)$. Notice that thanks to Assumption 4.1, $\Phi_t(\cdot)$ is continuously differentiable, and hence the conditions of Proposition A.3 are respected.

Observing that

$$\Phi_t(y_1) - \Phi_t(y_2) = \nabla k(y_2)(y_1 - y_2) + \int_0^1 (\nabla k(y_2 + s(y_1 - y_2)) - \nabla k(y_2))(y_1 - y_2)\, ds$$

we may write

$$\begin{aligned} &\langle y_2^*, y_1 - y_2 \rangle \\ &= \langle v_2^*, \Phi_t(y_1) - \Phi_t(y_2) \rangle - \left\langle v_2^*, \int_0^1 (\nabla k(y_2 + s(y_1 - y_2)) - \nabla k(y_2))(y_1 - y_2)\, ds \right\rangle \\ &\le 0 + \beta \|v_2^*\| \, \|y_1 - y_2\|^2, \end{aligned} \tag{43}$$

where the constant

$$\beta := (1/2) \sup_{y \in \operatorname{co}(\operatorname{Rge} S)} \left\| \frac{\partial^2 k}{\partial y^2}(y) \right\|$$

is finite because of the boundedness assumption of $\frac{\partial^2 k}{\partial y^2}(\cdot)$. By (38), like in the proof of Lemma 4.2, for any $q \in \mathbb{B}_{\mathbb{R}^m}$ there exist $q' \in \mathbb{B}_{\mathbb{R}^n}$ and $p \in \mathbb{R}^m_+$ such that

$$\rho \langle v_2^*, q \rangle = \langle v_2^*, \nabla k(z_2) q' + p \rangle \le \langle y_2^*, q' \rangle \le \|y_2^*\|$$

and hence $\|v_2^*\| \le \frac{1}{\rho} \|y_2^*\|$. Using this and (43) we obtain

$$\langle y_1^* - y_2^*, y_1 - y_2 \rangle \ge -\frac{\beta}{\rho} (\|y_1^*\| + \|y_2^*\|) \|y_1 - y_2\|^2.$$

From [36, Theorem 4.1] (which is recorded in Section A.2) it follows that, for

$$r := \rho\left(\sup_{y\in \mathrm{co}\,(\mathrm{Rge}\,S)}\left\|\frac{\partial^2 k}{\partial y^2}(y)\right\|\right)^{-1}, \tag{44}$$

the set $S(t)$ is r-prox-regular. So, [16, Theorem 1] may be applied, and (c) is proved. Under the additional growth condition, it is enough to handle with Theorem A.4 and its related comments in Section A.5 to conclude that (c) holds. The proof is then complete. □

It is noteworthy that the right-hand-side in (44) is a conservative estimation of the prox-regularity of the set $S(t)$, in the sense that $S(t)$ may be prox-regular with a larger r. However it is clear that in the context of Section 3 one recovers convexity since (44) gives $r = +\infty$ when $S(t)$ is finitely represented by affine functions.

4.3.2. The *rcbv* case (state jumps)

Let us now deal with the case where $u(\cdot)$ is a locally *rcbv* function. From the analysis of the linear case in Section 3, it is expected that the state $x(\cdot)$ will have jumps, so that the measure differential inclusion framework in (20) will be needed.

Theorem 4.4. *Consider the system in* (34) *and suppose that* $a(\cdot)$, $e(\cdot)$ *are continuous,* $g(\cdot)$ *is locally Lipschitz continuous, and the mapping* $\tilde{e}(\cdot,\cdot)$ *in* (40) *is bounded. Suppose also that Assumption* 4.1 *and inclusion* (38) *hold and that* $\frac{\partial^2 k}{\partial z^2}(\cdot)$ *is bounded on the convex hull* $\mathrm{co}\,(\mathrm{Rge}\,S)$ *of* $\mathrm{Rge}\,S$, *where* $k(\cdot)$ *is as in Lemma* 4.2. *Let* $u(\cdot)$ *be locally* rcbv, *and* $z_0 \in S(0)$. *Then the following assertions hold.*

(a) *For* $r = \rho\big(\sup_{y\in\mathrm{co}\,(\mathrm{Rge}\,S)}\|\frac{\partial^2 k}{\partial y^2}(y)\|\big)^{-1}$, *all the sets* $S(t)$ *are* r*-prox-regular. If the inequality* $\sup_{s\in]0,+\infty[}\mu(\{s\}) < \frac{r}{2}$ *is satisfied (where* μ *denotes the differential measure associated with* $var_C(\cdot)$*), then there exists some* $T > 0$ *such that the perturbed measure differential inclusion corresponding to* (40)*, i.e.,*

$$\begin{cases} -dz \in N(S(t); z(t)) - (\tilde{h}(z(t)) + \tilde{e}(z(t), u(t)))\,d\lambda \\ z(0) = z_0 \in S(0) \end{cases} \tag{45}$$

has at least one rcbv *solution on* $[0, T]$ *(recalling that* λ *denotes the one-dimensional Lebesgue measure).*

(b) *If in addition,* $a(\cdot)$ *and* $e(\cdot, u(t))$ *are Lipschitz continuous on bounded subsets of* $\mathbb{R}^n$ *(the dependence of the Lipschitz constant of* $\tilde{e}(\cdot, u(t))$ *being* $\mathrm{L}^\infty_{\mathrm{loc}}$*), if* $\sup_{s\in]0,+\infty[}\mu(\{s\}) < \frac{r}{4}$, *and if the mapping* $\tilde{h}$ *satisfies for some constant* $\beta \geq 0$ *the linear growth condition* $\|\tilde{h}(y)\| \leq \beta(1 + \|y\|)$ *for all* $y \in \mathbb{R}^n$, *then one has one and only one locally* rcbv *solution on* $[0, +\infty[$.

Proof. As already seen in Theorem 4.3, all the sets $S(t)$ are nonempty. The proof of the r-prox-regularity of these sets $S(t)$ is exactly the same as in the second part of the proof of Theorem 4.3. Further, the first part of the same proof gives that the set-valued mapping $S(\cdot)$ is locally *rcbv*. Fix $\tau > \|z_0\|$ and observe that, for $f(t, y) := -\tilde{h}(y) - \tilde{e}(y, u(t))$, the mapping $\tilde{f}(\cdot,\cdot)$ defined in (41) is bounded on $[0, \tau] \times \mathbb{R}^n$ (since

$\tilde{e}(\cdot,\cdot)$ is bounded by assumption and $\tilde{h}(\cdot)$ is continuous), measurable with respect to its first variable and continuous with respect to its second variable. We may then apply Theorem A.5 in place of [40, Theorem 4.4] and we obtain an *rcbv* solution $z(\cdot)$ on $[0,\tau]$ of the perturbed measure differential inclusion corresponding to (42), say

$$-dz \in N(S(t); z(t)) + \tilde{f}(t, z(t))d\lambda. \tag{46}$$

The right continuity of $z(\cdot)$ at 0 combined with the inequality $\|z_0\| < \tau$ yields some positive number $T < \tau$ (as in Theorem 4.3) such that $\|z(t)\| \le \tau$ for all $t \in [0,T]$. So we have $\tilde{f}(t,z(t)) = f(t,z(t))$ for all $t \in [0,T]$ and hence $z(\cdot)$ is also a solution on $[0,T]$ of the measure differential inclusion (45). Assertion (a) of the theorem is then established.

Observe now that the existence result over $[0,+\infty[$ under the assumptions in (b) follows from the comments after Theorem A.5. Let us deal with the uniqueness. Fix any $\theta > 0$.

Step 1. Let us first establish an estimation. Let $z(\cdot)$ be an *rcbv* solution of (45) on $[0,\theta]$. Let μ be the Radon measure associated with the *rcbv* function $var_S(\cdot)$ and let $\nu := \mu + \lambda$. By (19) we know that

$$-\frac{dz}{d\nu}(t) - f(t,z(t))\frac{d\lambda}{d\nu}(t) \in N(S(t); z(t)) \quad \nu - \text{a.e. } t \in [0,\theta], \tag{47}$$

where $\frac{d\lambda}{d\nu}(\cdot)$ and $\frac{dz}{d\nu}(\cdot)$ denote the differential measures with respect to ν of the measure λ and the vector measure dz associated with the *rcbv* mapping $z(\cdot)$. Note that by (ii) of Section 3.2.1 (resp. by the obvious absolute continuity of λ with respect to ν) the mapping $\frac{dz}{d\nu}(\cdot)$ (resp. $\frac{d\lambda}{d\nu}(\cdot)$) is the density relative to ν of dz (resp. λ).

Fix any $t \in]0,\theta]$ such that $\frac{dz}{d\nu}(t)$, $\frac{d\mu}{d\nu}(t)$, and $\frac{d\lambda}{d\nu}(t)$ exist and such that

$$\xi(t) := \frac{dz}{d\nu}(t) + f(t,z(t))\frac{d\lambda}{d\nu}(t) \neq 0. \tag{48}$$

We follow some ideas in the proof of Proposition 2.2 in [40]. Since the closed set $S(t)$ is prox-regular, its normal cone coincides with its Fréchet normal cone and hence (see the Appendix A.1) we have according to (47) that $-\frac{\xi(t)}{\|\xi(t)\|}$ is a Fréchet subgradient of the distance function $d(\cdot, S(t))$ at the point $z(t) \in S(t)$. Then for any $\varepsilon > 0$, for positive real number $s < t$ sufficiently close to t we have

$$\begin{aligned}\left\langle -\frac{\xi(t)}{\|\xi(t)\|}, z(s) - z(t)\right\rangle &\le d(z(s), S(t)) - d(z(t), S(t)) + \varepsilon\|z(s) - z(t)\| \\ &= d(z(s), S(t)) - d(z(s), S(s)) + \varepsilon\|z(t) - z(s)\| \\ &\le \mathrm{var}_S(t) - \mathrm{var}_S(s) + \varepsilon\|z(t) - z(s)\|,\end{aligned}$$

the equality above being due to the fact that $d(z(t), S(t)) = 0 = d(z(s), S(s))$ because $z(\tau) \in S(\tau)$ for all τ, and the second inequality being due to the definition of the variation function var_S. Therefore for s as above, observing that $\nu(]s,t]) \ge \lambda(]s,t]) > 0$, we obtain

$$\left\langle \frac{\xi(t)}{\|\xi(t)\|}, \frac{z(t) - z(s)}{\nu([s,t])}\right\rangle \le \frac{\mathrm{var}_S(t) - \mathrm{var}_S(s)}{\nu([s,t])} + \varepsilon\left\|\frac{z(t) - z(s)}{\nu([s,t])}\right\|,$$

which yields for $s \uparrow t$ acccording to (17)

$$\left\langle \frac{\xi(t)}{\|\xi(t)\|}, \frac{dz}{d\nu}(t) \right\rangle \leq \frac{d\mu}{d\nu}(t) + \varepsilon \left\| \frac{dz}{d\nu}(t) \right\|$$

and hence

$$\left\langle \xi(t), \frac{dz}{d\nu}(t) \right\rangle \leq \|\xi(t)\| \frac{d\mu}{d\nu}(t).$$

By (48) the latter entails for any $t \in]0, \theta]$ such that $\frac{dz}{d\nu}(t)$, $\frac{d\mu}{d\nu}(t)$, and $\frac{d\lambda}{d\nu}(t)$ exist that

$$\langle \xi(t), \xi(t) \rangle \leq \langle \xi(t), f(t, z(t)) \rangle \frac{d\lambda}{d\nu}(t) + \|\xi(t)\| \frac{d\mu}{d\nu}(t),$$

thus

$$\|\xi(t)\| \leq \|f(t, z(t))\| \frac{d\lambda}{d\nu}(t) + \frac{d\mu}{d\nu}(t). \tag{49}$$

Step 2. Let $z_1(\cdot)$ and $z_2(\cdot)$ be two *rcbv* solutions of (45) on $[0, +\infty[$ and hence their restrictions to $[0, \theta]$ are solutions over the interval $[0, \theta]$ with initial condition $z_i(0) = z_0$ for $i = 1, 2$. Put

$$\xi_i(\cdot) := \frac{dz_i}{d\nu}(\cdot) + f(\cdot, z_i(\cdot)) \frac{d\lambda}{d\nu}(\cdot)$$

for $i = 1, 2$ and fix any $t \in]0, \theta]$ such that $\frac{dz_i}{d\nu}(t)$, $\frac{d\mu}{d\nu}(t)$, and $\frac{d\lambda}{d\nu}(t)$ exist. According to the r-prox regularity of $S(t)$ we have

$$\langle \xi_1(t) - \xi_2(t), z_1(t) - z_2(t) \rangle \leq \frac{1}{r} \left(\|\xi_1(t)\| + \|\xi_2(t)\| \right) \|z_1(t) - z_2(t)\|^2. \tag{50}$$

Further since the mappings $z_i(\cdot)$ are of bounded variation on $[0, \theta]$ we have $\|z_i(s) - z_i(0)\| \leq \mathrm{var}_{z_i}([0, \theta])$ for all $s \in [0, \theta]$ and hence both mappings $z_1(\cdot)$ and $z_2(\cdot)$ are bounded on $[0, \theta]$. It then follows from the boundedness asssumption of $\tilde{e}$ and the linear growth condition of $\tilde{h}$ that there exists some constant β_1 (independent of s) such that $\|f(s, z_1(s))\| + \|f(s, z_2(s))\| \leq \beta_1$ for all $s \in [0, \theta]$. According to (50) we obtain for t as above (and hence for ν-a.e. $t \in]0, \theta]$)

$$\left\langle \frac{dz_1}{d\nu}(t) - \frac{dz_2}{d\nu}(t), z_1(t) - z_2(t) \right\rangle \leq \frac{1}{r} \left(\|\xi_1(t)\| + \|\xi_2(t)\| + r\beta_1 \frac{d\lambda}{d\nu}(t) \right) \|z_1(t) - z_2(t)\|^2. \tag{51}$$

Step 3. Put $g(s) := \frac{2}{r}(\|\xi_1(s)\| + \|\xi_2(s)\| + r\beta_1 \frac{d\lambda}{d\nu}(s))$. By (51) and Proposition 3.3 we have for any $t \in [0, \theta]$

$$\|z_1(t) - z_2(t)\|^2 \leq \int_{]0,t]} g(s) \|z_1(s) - z_2(s)\|^2 \, d\nu(s). \tag{52}$$

On the other hand, we observe by (49) that

$$\begin{aligned}
\|\xi_i(t)\| \, \nu(\{t\}) &\leq \|f(t, z_i(t))\| \frac{d\lambda}{d\nu}(t) \, \nu(\{t\}) + \frac{d\mu}{d\nu}(t) \, \nu(\{t\}) \\
&= \int_{\{t\}} \|f(s, z_i(s))\| \frac{d\lambda}{d\nu}(s) \, d\nu(s) + \int_{\{t\}} \frac{d\mu}{d\nu}(s) \, d\nu(s),
\end{aligned}$$

and hence using the absolute continuity of λ and μ with respect to ν we obtain

$$\|\xi_i(t)\|\,\nu(\{t\}) \leq \int_{\{t\}} \|f(s, z_i(s))\|\,d\lambda(s) + \int_{\{t\}} d\mu(s) = \mu(\{t\}), \tag{53}$$

where the equality follows from $\lambda(\{t\}) = 0$. Since we also have $r\beta_1 \frac{d\lambda}{d\nu}(t)\nu(\{t\}) = r\beta_1 \int_{\{t\}} d\lambda(s) = 0$, we see by (53) that for any t as in *Step 2.* we have

$$0 \leq g(t)\nu(\{t\}) \leq \frac{4}{r} \sup_{s\in]0,\theta]} \mu(\{s\} =: \sigma < 1,$$

where the inequality $\sigma < 1$ follows from the assumption $\sup_{s\in]0,\theta]} \mu(\{s\} < r/4$. We may then apply Lemma 3.4 to obtain for all $t \in [0, \theta]$

$$\|z_1(t) - z_2(t)\|^2 \leq 0, \quad \text{i.e. } z_1(t) = z_2(t).$$

So the uniqueness of the solution in $[0, +\infty[$ is established and the proof of the theorem is complete. $\square$

4.4. Examples

Example 4.5. Consider a nonlinear complementarity system of the form:

$$\begin{cases} \dot{x}(t) = -x^3(t) + \zeta(t) + (x^2(t) + 1)u(t) \\ 0 \leq w(t) = x(t) + \sin(u(t)) \perp \zeta(t) \geq 0 \end{cases} \tag{54}$$

with $x(t) \in \mathbb{R}$ and $w(t) \in \mathbb{R}$. Let us choose $V(x) = \frac{1}{2}x^2$. Since $b = 1$ one obtains $h(x) = x$, so that $z = x$. One can check that the uncontrolled system in (54) is dissipative with storage function equal to $V(x)$ since

$$V(x(t)) - V(x(0)) = \int_0^t x(s)\zeta(s)ds - \int_0^t x^4(s)ds = -\int_0^t x^4(s)ds$$

where the last equality holds since $x(t)\zeta(t) = 0$. This secures that Assumption 4.1 holds (see for instance [9, Lemma 4.84]). Applying the above transformation we get

$$-\dot{z}(t) - z^3(t) + (z^2(t) + 1)u(t) \in \partial\psi_{[-\sin(u(t)),+\infty[}(z(t)).$$

where obviously $S(t) = [-\sin(u(t)), +\infty[$ is non empty. Since the perturbation is locally Lipschitz continuous, we obtain that for any initial condition $z_0 \in [-\sin(u(0)), +\infty[$ there exists a local solution which is absolutely continuous and it is unique, provided $u(\cdot)$ is itself locally absolutely continuous. Since the sets $S(t)$ are convex ($r = +\infty$) then Theorem 4.4(a) applies when $u(\cdot)$ is locally *rcbv*.

Example 4.6. Consider a nonlinear complementarity system of the form

$$\begin{cases} \dot{x}(t) = Ax(t)x^T(t)Gx(t) + B\zeta(t) + e(x(t), u(t)) \\ 0 \leq w(t) = Cx(t) + g(u(t)) \perp \zeta(t) \geq 0 \end{cases} \tag{55}$$

where $x(t) \in I\!R^n$, $w(t) \in I\!R^m$, $g(\cdot)$ and $e(\cdot)$ have the required regularity properties. Let $PB = C^T$ for some $P = P^T > 0$, hence (35) holds with $V(x) = \frac{1}{2}x^T Px$. The state vector change is given by $z = Rx$ with R a symmetric positive definite square root of P. It is assumed that $K(t) = \{x \in I\!R^n \mid Cx + g(u(t)) \geq 0\}$ is non empty, which is guaranteed if, for instance, C has rank m. Here $k(y) = c \circ h^{-1}(y)$ is simply equal to $k(y) = CR^{-1}y$ with $h(x) = Rx$, and $r = +\infty$ in Theorem 4.3(b) and Theorem 4.4(a). So Theorems 4.3 and 4.4(a) apply.

5. Conclusion

In this paper some results in [20, 5] on existence and uniqueness of solutions of a class of nonsmooth autonomous dynamical systems, are extended to non-autonomous systems. A specific "input-output" property of the considered systems is used to perform a change of state vector, which allows us to transform the dynamics into a perturbed Moreau's sweeping process. Two cases are examined: when the vector field is linear and when it is nonlinear. Then the results of [16, 15, 40] are used to prove existence and uniqueness of absolutely continuous and of locally bounded variation solutions. Consequently nonlinear, nonsmooth systems with state jumps and possible accumulations of jumps (the Zeno phenomenon) are allowed. This work may also be seen as enlarging the studies on the relationships between various kinds of nonsmooth dynamical systems like differential inclusions, complementarity systems, projected systems, and variational inequalities, as initiated in [8, 22].

A. Appendix

A.1. Subdifferential and normal cones

The following notions extend the concepts of subgradient and normal cone of convex analysis, to non-convex functions and sets. We recall the definitions and results from [38, 29] which have been used in the previous sections.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function which is finite at x. A vector $v \in \mathbb{R}^n$ is a Fréchet subgradient of $f(\cdot)$ at x if for each $\varepsilon > 0$ there exists some neighborhood U of x such that $\langle v, x' - x\rangle \leq f(x') - f(x) + \varepsilon\|x' - x\|$ for all $x' \in U$. The set of all Fréchet subgradients of $f(\cdot)$ at x is the Fréchet subdifferential of $f(\cdot)$ at x. A limiting process leads to the Mordukhovich basic subdifferential of $f(\cdot)$ at x. A vector v is a *basic subgradient* of $f(\cdot)$ at x if there exist a sequence $(x_n, f(x_n))_n$ converging to $(x, f(x))$ and a sequence $(v_n)_n$ of Fréchet subgradients of $f(\cdot)$ at x_n converging to v. The set $\partial f(x)$ of all such basic subgradients is the (Mordukhovich) *basic subdifferential* of $f(\cdot)$ at x. When $f(x)$ is not finite, one puts $\partial f(x) = \emptyset$.

For a subset Q of $\mathbb{R}^n$, the Fréchet or basic subdifferential of its indicator function $\psi_Q(\cdot)$ at $x \in Q$ is the Fréchet or basic *normal cone* of Q at x. The basic normal cone is denoted by $N(Q; x)$. Recall that $\psi_Q(x') = 0$ if $x' \in Q$ and $\psi_Q(x') = +\infty$ otherwise.

Obviously the Fréchet normal cone of Q at $x \in Q$ is included in the basic normal cone $N(Q; x)$. When the inclusion is an equality, the set Q is said to be *normally regular* at x. Such a normal regularity entails in particular that both above normal cones coincide with the Clarke normal cone of Q at x, i.e., the closed convex hull of the basic normal cone. It is known (see [29]) that a vector v is a Fréchet subgradient of the distance function $d(\cdot, Q)$ at a point $x \in Q$, around which Q is closed, if and only if v a Fréchet normal of Q at x with $\|v\| \leq 1$. So, when Q is closed near $x \in Q$ and normally regular at x, one has $\partial d(\cdot, Q)(x) = N(Q; x) \cap \mathbb{B}_{\mathbb{R}^n}$.

If the epigraph in $\mathbb{R}^n \times \mathbb{R}$ of the function $f(\cdot)$, say $\operatorname{epi} f := \{(x', r) \in \mathbb{R}^n \times \mathbb{R} : f(x') \leq r\}$, is normally regular at $(x, f(x))$, one says that the function $f(\cdot)$ is *subdifferentially regular* at x. When $f(\cdot)$ is convex, it is subdifferentially regular at any point x where it is finite and $\partial f(x)$ coincides with the usual subdifferential in the sense of Convex Analysis. In summary, one has the following assertions:

- If the Fréchet and the basic Mordukhovich cones to a set Q are equal, then Q is normally regular.
- If Q is normally regular, then the Fréchet, the basic Mordukhovich and the Clarke cones are equal.
- If Q is $r-$prox-regular (see the definition in A.2 below), then the Fréchet, the basic Mordukhovich, and the Clarke cones are equal.
- However $r-$prox-regularity and normal regularity are different notions, in the sense that there are normally regular sets which are not prox-regular.

The concept of singular subdifferential $\partial^\infty f(x)$ of $f(\cdot)$ will be also needed. A vector $v \in \partial^\infty f(x)$ if and only if $(v, 0) \in N(\operatorname{epi} f; (x, f(x))$.

A.2. r-prox-regular set

Let H be a Hilbert space. A nonempty closed subset Q of H is r-prox-regular, for some $r \in]0, +\infty]$, provided the distance function $d(\cdot, Q)$ is Fréchet differentiable on the open tube $\{x \in H : 0 < d(x, Q) < r\}$. (Recall that $d(x, Q) = \inf\{\|x - y\| : y \in Q\}$). Closed convex sets and C^2-submanifolds are prox-regular. See, e.g., [36, 38] for several other examples as well as results of preservation of prox-regularity under operations. Roughly, in the finite dimensional setting, prox-regular sets may be non convex sets for which the projection onto the set is uniquely defined for points close enough to the set.

The following characterization, where $N(Q; \cdot)$ denotes the basic normal cone to Q, is extracted from Theorem 4.1 in [36]. The closed set Q is r-prox-regular if and only if for any $x_i \in Q$ $(i = 1, 2)$, the inequality

$$\langle v_1 - v_2, x_1 - x_2 \rangle \geq -\|x_1 - x_2\|^2$$

holds whenever $v_i \in N(Q; x_i)$ with $\|v_i\| < r$.

So, any closed convex subset of H is r-prox-regular with $r = +\infty$.

A.3. Some technical results

The next result is proved in [39, p. 760], [38, Example 9.35] (see also the previous result in [44, Theorem 1] concerning the statement below with $S = P \cap \{x \in \mathbb{R}^n : Ax - b = c\}$ and $S' = P \cap \{x \in \mathbb{R}^n : Ax - b' = c\}$, where $c \in \mathbb{R}^m$ is fixed and P is some fixed closed convex polyhedral subset of $\mathbb{R}^n$). For the ease of reading it is presented here as a lemma.

Lemma A.1. *Let $S = \{x \in \mathbb{R}^n : \ Ax - b \in K\}$, $S' = \{x \in \mathbb{R}^n : \ Ax - b' \in K\}$, where A is an $m \times n$ matrix and K is a closed convex polyhedral cone of $\mathbb{R}^m$ of the form $K = \{y \in \mathbb{R}^m : \ Dy \geq 0\}$ for some matrix D. Then there exists a constant δ depending only on A and D such that haus$(S, S') \leq \delta \|D\| \|b - b'\|$ whenever S and S' are non empty, that is $b, b' \in \operatorname{Rge} A - K$.*

The next two results are generalizations of the chain rule for the differentiation of composed nonsmooth functions [29, 37, 38]. The first one, for which we refer to [37], is when a convex function is composed with a linear mapping. The second one is when a lower semicontinuous function is composed with a smooth mapping.

Proposition A.2. *Let $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function, and $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. Assume that either the function f is polyhedral or for some x_0 with $Ax_0 \in \operatorname{dom} f$*

$$\operatorname{Rge} A - \mathbb{R}_+ (\operatorname{dom} f - Ax_0) \quad \textit{is a vector subspace of}\, \mathbb{R}^m, \tag{56}$$

where $\operatorname{dom} f := \{y \in \mathbb{R}^m : \ f(y) < +\infty\}$. *Then the subdifferential in the sense of convex analysis of the composite functional $f \circ A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by*

$$\partial(f \circ A)(x) = A^T \partial f(Ax), \ \ \forall\, x \in \mathbb{R}^n. \tag{57}$$

At several places of Section 4 the next result is used. Its part (a) is extracted from [29, Theorem 3.41] or [38, Theorem 10.6] and (b) is a consequence of (a).

Proposition A.3. *Suppose $g = f \circ F$ for a proper, lower semicontinuous function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, and a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ which is continuously differentiable at a point x where g is finite.*

(a) *If $f(\cdot)$ is subdifferentially regular at $F(x)$ and if*

$$\partial^\infty f(F(x)) \cap \operatorname{Ker} \nabla F(x)^T = \{0\},$$

then $g(\cdot)$ is subdifferentially regular at x, and

$$\partial g(x) = \nabla F(x)^T \partial f(F(x)). \tag{58}$$

(b) *In particular, the conclusions of (a) hold whenever f is convex and*

$$\operatorname{Rge} \nabla F(x) + \mathbb{R}_-(\operatorname{dom} f - F(x)) = \mathbb{R}^m. \tag{59}$$

A.4. Well-posedness of LCPs

Let us consider the linear complementarity problem (LCP)

$$0 \leq \zeta \perp M\zeta + q \geq 0 \tag{60}$$

where M is an $m \times m$ P-matrix. Then ζ is unique and is a continuous piecewise linear function of q (hence Lipschitz continuous) [35]. P-matrices are matrices with all their principal minors positive. They are full-rank, not necessarily symmetric. Positive definite (possibly non-symmetric) matrices are P-matrices. A symmetric P-matrix is positive definite.

A.5. Theorem 1 of [16]

For the ease of reading let us recall Theorem 1 of [16]. Let $I = [t_0, t_1]$, $t_0 < t_1$ be an interval of $\mathbb{R}$ and $S(\cdot)$ be a set-valued mapping from I to the Hilbert space H. It is assumed that:

(H1) For each $t \in I$, $S(t)$ is a nonempty closed subset of H which is r-prox-regular;

(H2) $S(t)$ varies in an absolutely continuous way, that is, there exists an absolutely continuous function $v(\cdot) : I \to \mathbb{R}$ such that for any $y \in H$ and $s, t \in I$ one has $|d(y, S(t)) - d(y, S(s))| \leq |v(t) - v(s)|$.

The notation $B[0,\eta]$ means the closed ball of radius η centered at 0.

Theorem A.4. *Let $f : I \times H \to H$ be a separately measurable map on I such that*

(i) *For every $\eta > 0$ there exists a non-negative function $k_\eta(\cdot) \in L^1(I,\mathbb{R})$ such that for all $t \in I$ and for any $(x,y) \in B[0,\eta] \times B[0,\eta]$ one has $||f(t,x) - f(t,y)|| \leq k_\eta(t)||x-y||$;*

(ii) *there exists a non-negative function $\beta(\cdot) \in L^1(I,\mathbb{R})$ such that, for all $t \in I$ and for all $x \in \bigcup_{s\in I} S(s)$, $||f(t,x)|| \leq \beta(t)(1+||x||)$.*

Then for any $x_0 \in S(t_0)$ the following perturbed sweeping process

$$\begin{cases} -\dot{x}(t) \in N(S(t),x(t)) + f(t,x(t)) & a.e.\ t \in I \\ x(t_0) = x_0 \in S(t_0) \end{cases} \tag{61}$$

has one and only one absolutely continuous solution $x(\cdot)$. This solution satisfies $||\dot{x}(t) + f(t,x(t))|| \leq (1+l)\beta(t) + |\dot{v}(t)|$ a.e. $t \in I$, and $||f(t,x(t))|| \leq (1+l)\beta(t)$ a.e. $t \in I$, where

$$l = ||x_0|| + \exp\left\{\int_I \beta(s)ds\right\} \int_I [2\beta(s)(1+||x_0||) + |\dot{v}(s)|]\, ds.$$

If $I = [t_0, +\infty)$, the function $v(\cdot)$ is locally absolutely continuous on I, and $k_\eta(\cdot)$ and $\beta(\cdot)$ are locally integrable on I, then it easily follows from the above theorem (applying it succesively to the intervals $[t_0, t_0+1]$, $[t_0+1, t_0+2]$ etc) that (59) has one and only one locally absolutely continuous solution $x(\cdot)$ on I.

A.6. Theorem 4.1 of [15]

The notation is the same as in the foregoing subsection. It is assumed that:

(H1) For each $t \in I$, $S(t)$ is a nonempty closed subset of H which is r-prox-regular;

(H2) The set-valued mapping $S(\cdot)$ is of right continuous bounded variation on I, i.e., it is of bounded variation on I and its variation function $var_S(\cdot)$ is right continuous on I.

The Radon measure associated with $var_S(\cdot)$ is denoted as μ, so that for any $s,t \in I$ with $s \leq t$ one has $|d(y,S(t)) - d(y,S(s))| \leq \mu(]s,t])$ for all $y \in H$.

Theorem A.5. *Let $F : I \times H \to H$ be a set-valued mapping with nonempty convex compact values such that*

(i) *for any $x \in H$, $F(\cdot,x)$ has a $\lambda-$measurable selection;*

(ii) *for all $t \in I$, $F(t,\cdot)$ is scalarly upper semicontinuous on H;*

(iii) *for some compact subset K of the unit ball of H and for some real number $\beta \geq 0$, we have $F(t,x) \subset \beta(1+||x||)K$ for all $(t,x) \in I \times H$.*

Assume that $\sup_{s\in(t_0,t_1]} \mu(\{s\}) < \frac{r}{2}$. Then for any $x_0 \in S(t_0)$, the following sweeping process on I with perturbation

$$\begin{cases} -dx \in N(S(t),x(t)) + F(t,x(t))d\lambda \\ x(t_0) = x_0 \end{cases} \tag{62}$$

has at least one rcbv *solution* $x(\cdot)$. *More precisely, if* $(\beta+1)(t_1-t_0) \leq \frac{1}{4}$, *setting* $l = 2\left(\mu(]t_0,t_1]) + ||x_0|| + 1\right)$, *for* $\nu = \mu + (l+1)(\beta+1)\lambda$, *there exists a* $\lambda-$*integrable map* $z : I \to H$ *such that, for* $\lambda-$*almost all* $t \in I$, $z(t) \in F(t,x(t))$ *and* $z(t) \in (l+1) - (\beta+1)\bar{co}(K \cup \{0\})$. *Moreover for* $\nu-$*almost all* $t \in I$,

$$\frac{dx}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t) \in -N(S(t), x(t)),$$

$$\left\|\frac{dx}{d\nu}(t) + z(t)\frac{d\lambda}{d\nu}(t)\right\| \leq 1$$

and

$$\left\|z(t)\frac{d\lambda}{d\nu}(t)\right\| \leq 1.$$

Observe that the assumption $\sup_{s\in(t_0,t_1]} \mu(\{s\}) < \frac{r}{2}$ automatically holds whenever the sets $S(t)$ are convex since in this case $r = +\infty$.

Suppose now that $I = [t_0, +\infty[$ and that in (iii) (instead of being a constant) $\beta(\cdot)$ is an $L^\infty_{loc}(I, \mathbb{R})$ function such that $F(t,x) \subset \beta(t)(1 + \|x\|)K$ for all $(t,x) \in [t_0, +\infty[\times H$. In that case, like in the comments after Theorem A.4, if $var_S(\cdot)$ is locally *rcbv* and if $\sup_{s\in]t_0,+\infty[} \mu(\{s\}) < \frac{r}{2}$, then the differential inclusion (62) has at least one locally *rcbv* solution $x(\cdot)$ on $[t_0, +\infty[$.

A.7. About the rank of C

Let $C \in \mathbb{R}^{3\times 2}$ be a real matrix with 3 rows and 2 columns, so that C has a rank ≤ 2 (and hence not equal to 3). The condition in (22) means that for all $x \in I\!R^3$ there exists $z \in \mathbb{R}^3_+$ and $v \in \mathbb{R}^2$ such that $Cv - z = x$. Consider $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{pmatrix}$, with $\min\{a,b\} < 0$, which has rank 2. Solving $Cv - z = x$ means finding $z_1 \geq 0$, $z_2 \geq 0$, $z_3 \geq 0$, $v_1 \in I\!R$, $v_2 \in I\!R$, such that $z_3 = -x_3 + ax_1 + bx_2 - az_1 - bz_2$, $v_1 = x_1 - z_1$, $v_2 = x_2 - z_2$. Obviously one can always choose $z_1 \geq 0$ and $z_2 \geq 0$ such that the first equality holds with $z_3 \geq 0$, and the problem is solved.

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